Journal of Applied Logic 8 (2010) 371–385

Contents lists available at ScienceDirect

Journal of Applied Logic

www.elsevier.com/locate/jal

A family of Gödel hybrid logics

Didier Galmiche *, Yakoub Salhi

LORIA – UHP Nancy 1, Campus Scientifique, BP 239, 54 506 Vandœuvre-lès-Nancy, France

A R T I C L E I N F O

Article history:
Available online 12 August 2010

Keywords:
Hybrid logic
Fuzzy logic
Gödel logic
Intermediate logics
Sequent-of-relations

A B S T R A C T

In this paper, we define a family of fuzzy hybrid logics that are based on Gödel logic. It is composed of two infinite-valued versions called GH_∞ and WGH_∞, and a sequence of finitary valued versions $(\mathsf{GH}_n)_{0<n<\infty}$. We define decision procedures for both WGH_∞ and $(\mathsf{GH}_n)_{0<n<\infty}$ that are based on particular sequents and on a set of proof rules dealing with such sequents. As these rules are strongly invertible the procedures naturally allow one to generate countermodels. Therefore we prove the decidability and the finite model property for these logics. Finally, from the decision procedure of WGH_∞, we design a sound and complete sequent calculus for this logic.

1. Introduction

Gödel logic and its finitary versions are logics between classical and intuitionistic logics (*intermediate logics*) with semantics based on linear Kripke models. It was introduced by Gödel and later axiomatized by Dummett in [12]. A Hilbert axiomatic system for this logic is obtained by adding $A \to B \vee B \to A$ to axioms of intuitionistic logic. Gödel logic is one of the most studied intermediate logics because it has been recognized as a *fuzzy logic* [17]. One of the interests of its finitary versions is that the countermodel search problem has been characterized as resource use bounding logics for a particular process calculus [21]. In addition a characterization of validity based on particular bi-colored graphs has been proposed for this logic [15].

We know that in the possible world semantics (Kripke semantics) of modal logics, any formula is either true or false at any world (2-valued modal logic) [7,10]. A *fuzzy modal logic* is a combination of a fuzzy logic and a modal logic such that a formula at a given world may have a truth value other than true and false (many-valued modal logic) [17]. In the literature, there exist various versions of fuzzy modal logics like, for instance, the one based on basic fuzzy logic [18] and also the finite-valued modal logics given in [13,14,24]. In this work we are interested in the fuzzy modal logics based on Gödel logic and its finitary versions [9,13,14,18]. Let us recall that hybrid logics were introduced in order to express this relativity of truth to the worlds of a model [1,6]. They are obtained by adding to modal logics a new kind of propositional symbols called *nominals*, and moreover, by adding a new operator, called *satisfaction operator*, that allows us to jump to the world named by a nominal.

In this paper, we define a family of fuzzy hybrid logics based on Gödel logic and its finitary versions. One of the motivations of such logics is the connection between hybrid logics and description logics [16] and their ability to reason about imprecise knowledge. The new logics presented here are obtained from the Gödel modal logics proposed in [9,13,14,18]. Our approach is similar to the one used to introduce the many-valued hybrid logic MVHL [19], where the Kripke models take values in a fixed finite Heyting algebra. The tableau system of MVHL essentially comes from the finiteness of the Heyting algebra and from the use of a language containing constants for all the truth values. In our family of fuzzy hybrid logics, there are logics built over infinite algebras and our language does not contain constants for the truth values.

* Corresponding author.
E-mail address: didier.galmiche@loria.fr (D. Galmiche).

1570-8683/$ – see front matter
doi:10.1016/j.jal.2010.08.008

We consider Kripke models where the accessibility relation between worlds is multi-valued. Propositional connectives are interpreted as usual in Gödel logic and modalities $\Box$ and $\Diamond$ are interpreted using the *infimum* and the *supremum*. The satisfaction operator and the nominals are interpreted as usual in hybrid logics. The family of our logics is composed of two infinite-valued versions, namely GH_∞ and WGH_∞, and a sequence of finite-valued versions $(\mathsf{GH}_n)_{0<n<\infty}$ (GH_n is the $(n+1)$-valued version). We have two infinite versions because of the two versions of infinite-valued Gödel logic in the literature, namely Gödel modal logic [9] and witnessed Gödel modal logic [18]. These two versions are different because in WGH_∞ we only consider the Kripke model where the infimum and the supremum correspond respectively to the minimum and the maximum.

After the presentation of our new logics, we study decidability and also proof-search in WGH_∞ and $(\mathsf{GH}_n)_{0<n<\infty}$. In this perspective we introduce a particular notion of sequent structure similar to the one given in [5]. Using this structure, we propose a set of proof rules and we prove that they are strongly sound and strongly invertible. It is important to notice that we define the same set of proof rules for both WGH_∞ and $(\mathsf{GH}_n)_{0<n<\infty}$. Then, we give decision procedures for WGH_∞ and $(\mathsf{GH}_n)_{0<n<\infty}$ that naturally allow to generate countermodels. They consist firstly in reducing, by a proof-search process using our set of proof rules, a sequent into a set of so-called irreducible sequents and then secondly in deciding these specific sequents by an appropriate procedure. In order to decide the irreducible sequents, we associate to any irreducible sequent a particular finite *bi-colored graph*, adapting a method developed for Gödel logic [22], and we give a characterization of validity based on the detection of particular chains in this graph. From these procedures we show that WGH_∞ and $(\mathsf{GH}_n)_{0<n<\infty}$ are decidable and also prove the finite model property. Finally we define a sequent calculus for WGH_∞ from our set of proof rules by adding an axiom and some structural rules.

2. Gödel modal logics

In this section, we propose a short description of finite and infinite versions of propositional Gödel modal logics [9,13,14,18].

We start by considering the family of propositional Gödel modal logics GM_n. The value n belongs to the set $\overline{\mathbb{N}}^* = \{1, 2, \ldots\} \cup \{\infty\}$ of strictly positive natural numbers augmented with a greatest element ∞.

The set of propositional *formulas* is inductively defined, starting from a set of propositional *variables* with an additional bottom constant $\bot$ denoting *absurdity* and using the connectives $\wedge$, $\vee$ and $\to$ and the modalities $\Box$ and $\Diamond$.

The semantics of GM_n is based on *fuzzy* Kripke models where the valuation at each world and also the accessibility relation between worlds are *many-valued*.

We note S_n the set of *truth-values* defined as follows:

$$S_n = \begin{cases} [0,1] & \text{if } n = \infty \\ \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} & \text{otherwise} \end{cases}$$

and we note $*$ and $\multimap$ the functions defined as follows:

$$a * b = \min\{a, b\}, \qquad a \multimap b = \begin{cases} 1 & \text{if } a \leqslant b \\ b & \text{otherwise} \end{cases}$$

Definition 2.1. A GM_n-Kripke model is a triple $\mathcal{M} = (W, R, V)$ in which W is a non-empty set of worlds, R is a function from $W \times W$ to S_n and V is a function from $W \times \mathsf{Var}$ to S_n (valuation).

The valuations are inductively extended to formulas as follows:

$V(w, \bot) = 0$;
$V(w, A \wedge B) = \min\{V(w, A), V(w, B)\}$;
$V(w, A \vee B) = \max\{V(w, A), V(w, B)\}$;
$V(w, A \supset B) = V(w, A) \multimap V(w, B)$;
$V(w, \Box A) = \inf\{R(w, w') \multimap V(w', A) \mid w' \in W\}$;
$V(w, \Diamond A) = \sup\{R(w, w') * V(w', A) \mid w' \in W\}$

where inf and sup denote respectively the infimum and the supremum.

A formula A is valid in $\mathcal{M} = (W, R, V)$, written $\mathcal{M} \vDash A$, iff $V(w, A) = 1$ for any $w \in W$. A formula A is valid in GM_n, written $\mathsf{GM}_n \vDash A$, iff it is valid in every GM_n-Kripke model. The negation $\neg$ is defined by $\neg A =_{def} A \supset \bot$. Note that the interdefinability between $\Box$ and $\Diamond$ given by $\Diamond A =_{def} \neg\Box\neg A$ breaks down in GM_n for $n > 1$.

We notice that for the sake of simplicity, we choose $S_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$. For example, we obtain the same logic if we take $S_n = \{0, r_1, r_2, \ldots, r_{n-1}, 1\}$ where $\{r_1, r_2, \ldots, r_{n-1}\}$ is a set of rational numbers with $0 < r_1 < r_2 < \cdots < r_{n-1} < 1$. It is because that the *Gödel algebra* having $\{0, r_1, r_2, \ldots, r_{n-1}, 1\}$ as underlying set is isomorphic to the one having $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ as underlying set. In fact, we can take any Gödel algebra having a set of size $n+1$ as underlying set. Let us recall that a Gödel algebra is a Heyting algebra satisfying prelinearity: $(x \to y) \vee (y \to x)$, i.e., for any elements x and y in the algebra, $x \leqslant y$ or $y \leqslant x$. For more details about Gödel algebras we refer to [17].

1) $A \supset (B \supset A)$	8) $(A \supset B) \supset ((C \supset A) \supset (C \supset B))$	
2) $(A \wedge B) \supset A$	9) $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$	
3) $(A \wedge B) \supset B$	10) $((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$	$\dfrac{A \supset B \quad A}{B}\ [MP]$
4) $A \supset (B \supset (A \wedge B))$	11) $(A \supset (B \supset C)) \supset ((A \wedge B) \supset C)$	
5) $\bot \supset A$	12) $((C \supset A) \wedge (C \supset B)) \supset (C \supset (A \wedge B))$	
6) $A \supset (A \vee B)$	13) $(A \supset (A \supset B)) \supset (A \supset B)$	
7) $B \supset (A \vee B)$	14) $(A \supset B) \vee (B \supset A)$	

Fig. 1. An axiomatization of Gödel logic.

In the particular case of GM_∞, Hilbert axiomatic system of the $\Box$-free and $\Diamond$-free fragments of this logic are provided in [9]. They are obtained by extending the standard axiomatization of the Gödel logic (Fig. 1) with the following axioms and rules:

$K_\Box$: $\Box(A \supset B) \supset (\Box A \supset \Box B)$	$D_\Diamond$: $\Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B)$
$Z_\Box$: $\neg\neg\Box A \supset \Box\neg\neg A$	$Z_\Diamond$: $\Diamond\neg\neg A \supset \neg\neg\Diamond A$
	$F_\Diamond$: $\neg\Diamond\bot$
$\dfrac{A}{\Box A}\ [RN_\Box]$	$\dfrac{A \supset B}{\Diamond A \supset \Diamond B}\ [RN_\Diamond]$

Clearly, in the GM_n-Kripke models with $n \neq \infty$, the infimum and the supremum functions, used in the definition of the extension of the valuations to formulas, correspond respectively to the minimum and the maximum.

This is not the case in all the GM_∞-Kripke models. Indeed, the value of $V(w, \Box A)$ (resp. $V(w, \Diamond A)$) may not be *witnessed* by the value of $R(w, w') \twoheadrightarrow V(w', A)$ (resp. $R(w, w') * V(w', A)$) for any world w'. For example the set $S = \{x \mid x > 0 \text{ and } x \in [0, 1]\}$ has 0 as infimum but $0 \notin S$. In fact, there is another Gödel modal logic obtained by restricting the validity notion to the models where every infimum (resp. supremum) is a minimum (resp. maximum). It is called the witnessed Gödel modal logic WGM_∞ (see [18]).

Definition 2.2. A WGM_∞-Kripke model $\mathcal{M} = (W, R, V)$ is a GM_∞-Kripke model where for all modal formulas $\Box A$ (resp. $\Diamond A$) and $w \in W$, we have $V(w, \Box A) = R(w, w') \twoheadrightarrow V(w', A)$ (resp. $V(w, \Diamond A) = R(w, w') * V(w', A)$) for a world $w' \in W$.

Proposition 2.3. *$\Box\neg\neg A \supset \neg\neg\Box A$ is valid in* WGM_∞.

Proof. We suppose that $\Box\neg\neg A \supset \neg\neg\Box A$ is not valid in WGM_∞. Let $\mathcal{M} = (W, R, V)$ be a countermodel of $\Box\neg\neg A \supset \neg\neg\Box A$. Then, there exists $w \in W$ such that $V(w, \Box\neg\neg A \supset \neg\neg\Box A) < 1$. Thus, $V(w, \Box\neg\neg A) > V(w, \neg\neg\Box A)$ holds. We have $V(w, \neg\neg\Box A) = (V(w, \Box A) \twoheadrightarrow 0) \twoheadrightarrow 0$ and since $V(w, \neg\neg\Box A) < 1$, $V(w, \Box A) = 0$ holds. Thus, there exists $w_0 \in W$ such that $R(w, w_0) \twoheadrightarrow V(w_0, A) = 0$ ($V(w_0, A) = 0$). We have $V(w, \Box\neg\neg A) = \min\{R(w, w') \twoheadrightarrow ((V(w', A) \twoheadrightarrow 0) \twoheadrightarrow 0)\} \geqslant 0$. Since $R(w, w_0) \twoheadrightarrow ((V(w_0, A) \twoheadrightarrow 0) \twoheadrightarrow 0) = 0$, $V(w, \Box\neg\neg A) = 0$ holds. As $V(w, \Box\neg\neg A) > V(w, \neg\neg\Box A)$, we get a contradiction. □

Proposition 2.4. *$\Box\neg\neg A \supset \neg\neg\Box A$ is not valid in* GM_∞.

Proof. A countermodel $\mathcal{M} = (W, R, V)$ of $\Box\neg\neg A \supset \neg\neg\Box A$ in GM_∞ is given in [9], with $W = \mathbb{N}$, for all $n, m \in W$, $R(n, m) = 1$ and for all $n \in W$, $V(n, p) = \frac{1}{n+1}$. □

Let L_1 and L_2 be two logics having the same syntax. $L_1 \subset L_2$ means that the set of valid formulas in L_1 is a subset of the set of valid formulas in L_2.

Theorem 2.5. $\mathrm{GM}_\infty \subset \mathrm{WGM}_\infty$.

Proof. WGM_∞ is a restriction of the class of models of GM_∞ and the result is deduced from Propositions 2.3 and 2.4. □

Each of the modal logics listed above except GM_∞ has the finite model property and is decidable [9,13,14,18]. We recall that a logic has a finite model property if and only if a formula in this logic is not valid then there exists a finite countermodel of this formula. The logic GM_∞ does not have the finite model property, however its $\Box$-free fragment has this property [9].

Gödel modal logics presented in this section correspond to minimal modal logics K. There exists a way to extend these logics with conditions on the accessibility relation [9]. For example, a GM_∞-Kripke modal $\mathcal{M} = (W, R, V)$ is *reflexive* if $R(w, w) = 1$ for all $w \in W$; it is *transitive* if $R(w, w') \times R(w', w'') \leqslant R(w, w'')$ for all $w, w', w'' \in W$; and it is *symmetric* if $R(w, w') = R(w', w)$ for all $w, w' \in W$.

3. Gödel hybrid logics

Hybrid logics are logics obtained by adding to modal logics a new kind of propositional symbols, called *nominals*, which are used to refer to specific worlds in a model. Moreover, new operators having many nice logical properties and called the *satisfaction operators* are added. They allow us to jump to the worlds named by nominals. For more details about hybrid logics see [1,6].

In this paper, we propose a family of fuzzy hybrid logics using the same formulas of classical hybrid logic and based on the Gödel modal logics presented previously. These logics are denoted by GH_n and WGH_∞ with $n \in \overline{\mathbb{N}}^*$. Our approach is similar to the one used to introduce the many-valued hybrid logic MVHL [19], where the Kripke models take values in a fixed finite Heyting algebra. Let us note that the tableau system of MVHL is based on the finiteness of the Heyting algebra and the use of a language containing constants for all the truth values. In our family of fuzzy hybrid logics, there are logics built over infinite algebras. Moreover, our language does not contain constants for the truth values.

Let Prop be a countably set of propositional symbols and Nom be a countably set of nominals, disjoint from Prop. We use $p, q, r, \ldots$ to range over Prop; and $a, b, c, \ldots$ to range over nominals.

The formulas of GH_n and WGH_∞ are given by the following grammar:

$$\mathcal{F} ::= p \mid a \mid \bot \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \supset \mathcal{F} \mid \Box\mathcal{F} \mid \Diamond\mathcal{F} \mid a : \mathcal{F}$$

$Nom(S)$ denotes the set of nominals that are in the syntactic object S.

Now, we give the semantics for each Gödel hybrid logic by using the same notion of Kripke model as defined in Section 2. A GH_n-Kripke model (resp. WGH_∞-Kripke model) is a GM_n-Kripke model (resp. WGM_∞-Kripke model).

Definition 3.1. Given a model $\mathcal{M} = (W, R, V)$, an $\mathcal{M}$-assignment is a function which assigns to each nominal a world in W. The extension of the valuation V w.r.t. the $\mathcal{M}$-assignment g, denoted V_g, is defined inductively as follows:

$$\begin{aligned}
&V_g(w, \bot) = 0;\\
&V_g(w, A \wedge B) = \min\{V_g(w, A), V_g(w, B)\};\\
&V_g(w, A \vee B) = \max\{V_g(w, A), V_g(w, B)\};\\
&V_g(w, A \supset B) = V_g(w, A) \twoheadrightarrow V_g(w, B);\\
&V_g(w, \Box A) = \inf\{R(w, w') \twoheadrightarrow V_g(w', A) \mid w' \in W\};\\
&V_g(w, \Diamond A) = \sup\{R(w, w') * V_g(w', A) \mid w' \in W\};\\
&V_g(w, a) = \begin{cases} 1 & \text{if } g(a) = w\\ 0 & \text{otherwise;}\end{cases}\\
&V_g(w, a : A) = V_g(g(a), A).
\end{aligned}$$

Clearly, each Gödel hybrid logic is conservative over its corresponding version of Gödel modal logic and of Gödel non-modal logic.

It is known that Gödel logic and its finitary versions are recognized as intermediate logics (between classical and intuitionistic logics). Now, we prove that our Gödel hybrid logics have such a property: they are between classical hybrid logic and intuitionistic hybrid logic. Let us note that IHL, studied by Braüner and de Paiva in [8], denotes the set of formulas valid in intuitionistic hybrid logic and CHL denotes the set of formulas valid in classical hybrid logic.

As in the case of classical and intuitionistic logics [2], we define an *intermediate hybrid logic* as a set of formulas $\mathcal{L}$ satisfying $\mathsf{IHL} \subseteq \mathcal{L} \subseteq \mathsf{CHL}$.

Theorem 3.2. *The following strictly decreasing sequence holds*:

$$\mathsf{CHL} = \mathsf{GH}_1 \supset \mathsf{GH}_2 \supset \cdots \supset \mathsf{GH}_n \supset \cdots \supset \mathsf{WGH}_\infty \supset \mathsf{GH}_\infty \supset \mathsf{IHL}$$

Proof. We start by proving that for all $n \in \mathbb{N}$, $\mathsf{GH}_n \supset \mathsf{GH}_{n+1}$ holds.

GH_n is a restriction of the class of models of GH_{n+1} in the sense that the GH_n-Kripke models are defined over a set of $n+1$ truth-values but the GH_{n+1}-Kripke models are defined over a set of $n+2$ truth-values. Thus, $\mathsf{GH}_n \supseteq \mathsf{GH}_{n+1}$ holds. Since GH_n and GH_{n+1} are conservative over respectively $n+1$-valued and $n+2$-valued Gödel non-modal logics and the second one is strictly included in the first one, $\mathsf{GH}_n \supset \mathsf{GH}_{n+1}$ holds.

$\mathsf{WGH}_\infty \supset \mathsf{GH}_\infty$ is proved from Theorem 2.5 and the fact that WGH_∞ and GH_∞ are respectively conservative over WGM_∞ and GM_∞.

Now, we prove that for all $n \in \overline{\mathbb{N}}^*$ such that $n \neq \infty$ we have $\mathsf{GH}_n \supset \mathsf{WGH}_\infty$. Since in any GH_n-Kripke model, with $n \neq \infty$, the infimum and the supremum correspond respectively to the minimum and the maximum, we have GH_n is a restriction of the class of models of WGH_∞. Thus $\mathsf{GH}_n \supseteq \mathsf{WGH}_\infty$ holds. Since WGH_∞ is conservative over the infinite version of Gödel non-modal logic and this version is strictly included in any finite version, $\mathsf{GH}_n \supset \mathsf{WGH}_\infty$ holds.

Finally, to prove that $\mathsf{GH}_\infty \supset \mathsf{IHL}$, we first prove $\mathsf{GH}_\infty \supseteq \mathsf{IHL}$ by showing the soundness of all the rules of the natural deduction system of IHL [8] in GH_∞. For instance, consider the rule of $[\Box_E]$:

$$\frac{W;[G \mid C \lhd a:A] \qquad W;[G \mid C \lhd a:B]}{W;[G \mid C \lhd a:A \wedge B]}\,[\wedge_R] \qquad \frac{W;[G \mid a:A \lhd C \mid a:B \lhd C]}{W;[G \mid a:A \wedge B \lhd C]}\,[\wedge_L]$$

$$\frac{W;[G \mid a:A \lhd C] \qquad W;[G \mid a:B \lhd C]}{W;[G \mid a:A \vee B \lhd C]}\,[\vee_L] \qquad \frac{W;[G \mid C \lhd a:A \mid A \lhd a:B]}{W;[G \mid C \lhd a:A \vee B]}\,[\vee_R]$$

$$\frac{W;[G \mid a:B < a:A] \qquad W;[G \mid a:B < C]}{W;[G \mid a:A \supset B < C]}\,[\supset_L^<]$$

$$\frac{W;[G \mid a:A \leqslant a:B \mid C < a:B] \qquad W;[G \mid C < \top]}{W;[G \mid C < a:A \supset B]}\,[\supset_R^<]$$

$$\frac{W;[G \mid \top \leqslant C \mid a:B < a:A] \qquad W;[G \mid a:B \leqslant C]}{W;[G \mid a:A \supset B \leqslant C]}\,[\supset_L^\leqslant]$$

$$\frac{W;[G \mid a:A \leqslant a:B \mid C \leqslant a:B]}{W;[G \mid C \leqslant a:A \supset B]}\,[\supset_R^\leqslant]$$

Fig. 2. Logical rules for GH_n and WGH_∞.

$$\frac{a:\Box A \qquad a:\Diamond e}{e:A}\,[\Box_E]$$

We suppose that $e:A$ has a countermodel in GH_∞. Then there are a GH_∞-Kripke model $\mathcal{M}=(W,R,V)$, an $\mathcal{M}$-assignment g and $w_0 \in W$ such that $V_g(w_0, A) < 1$. If $V_g(w_0, a:\Diamond e) = 1$ then $R(g(a), g(e)) = 1$ and $V_g(g(a), \Box A) < 1$. Otherwise, we have $V_g(w_0, a:\Diamond e) < 1$. Thus, $[\Box_I]$ is sound in GH_∞. Since IHL and GH_∞ are conservative over respectively intuitionistic and infinite-valued version of Gödel logic and the first one is strictly included in the second one, $\mathrm{GH}_\infty \supset \mathrm{IHL}$ holds. $\Box$

4. Proof rules

In this section, we introduce a set of proof rules that are strongly sound and strongly invertible for the finitary versions $(\mathrm{GH}_n)_{0<n<\infty}$ and WGH_∞. We split this set rules in three parts: the first one (see Fig. 2) contains the *logical rules* which decompose the formulas of the form $a:A \otimes B$ where $\otimes \in \{\wedge, \vee, \supset\}$; the second one (see Fig. 3) contains the *modal rules* applied to the formulas of the form $a:\boxtimes A$ where $\boxtimes \in \{\Box, \Diamond\}$ and A is a formula; and the third one (see Fig. 4) contains *satisfaction rules* applied to formulas of the form $a:b$. From now on, if we note GH_n, it means that we are only talking about the finitary versions.

Let us remind that the hypersequent structure $\Gamma_1 \vdash \Delta_1 \mid \cdots \mid \Gamma_k \vdash \Delta_k$ has been introduced as a generalization of Gentzen's sequent [3,4]. It is a multiset of sequents, called components, with "|" denoting a disjunction at the meta-level. Here, we are interested in a variant of the hypersequent structure called *sequent-of-relations* which is a nice structure for proof-search in Gödel logic [5]. Recently, a sequent calculus using this structure for the $\Diamond$-free fragment of GM_∞ has been defined in [23]. The structure that we use corresponds to the sequent-of-relations structure and an additional context, called *witness context*, in which some formulas are associated to nominals.

Definition 4.1. A *sequent* is a structure of the form

$$W;[A_1 \lhd_1 B_1 \mid A_2 \lhd_2 B_2 \mid \cdots \mid A_k \lhd_k B_k]$$

where W, called the *witness context*, is a set of pairs of the form $(c, a:A)$ with $c, a \in \mathrm{Nom}$ and A is of the form $\boxtimes B$ with $\boxtimes \in \{\Box, \Diamond\}$ and B a formula; and for $i = 1, 2, \ldots, k$, the sign $\lhd_i$ is either $\leqslant$ or $<$, and A_i and B_i are relative formulas, namely formulas of the form $a:A$, or constants ($\bot$ and $\top$).

We note *Form*($\mathcal{S}$), with $\mathcal{S}$ a sequent, the set of formulas in $\mathcal{S}$. Moreover, we define the notation $V_g(A)$ by

$$V_g(A) = \begin{cases} V_g(g(a), B) & \text{if } A \equiv a:B \\ 0 & \text{if } A \equiv \bot \\ 1 & \text{if } A \equiv \top \end{cases}$$

Definition 4.2. A sequent $\mathcal{S} = W;[A_1 \lhd_1 B_1 \mid \cdots \mid A_k \lhd_k B_k]$ is valid in the Kripke model $\mathcal{M} = (U, R, V)$ w.r.t. the $\mathcal{M}$-assignment g, written $\mathcal{M}, g \models \mathcal{S}$, iff if

- for all $(c, a:\Box A) \in W$, $V_g(g(a), \Box A) = R(g(a), g(c)) \mathbin{-\!\!\triangleright} V_g(g(c), A)$; and
- for all $(c, a:\Diamond A) \in W$, $V_g(g(a), \Diamond A) = R(g(a), g(c)) * V_g(g(c), A)$,

then there exists $i \in \{1, \ldots, k\}$ such that the inequality $V_g(A_i) \lhd_i V_g(B_i)$ holds.

$$\frac{W;[G \mid a:\Box A < C \mid b:A < a:\Diamond b] \qquad W;[G \mid a:\Box A < C \mid b:A < C]}{W;[G \mid a:\Box A < C]}\ [\Box_L^{<}(1)]$$

$$\frac{W;[G \mid a:\Box A \leqslant C \mid \top \leqslant C \mid b:A < a:\Diamond b] \qquad W;[G \mid a:\Box A \leqslant C \mid b:A \leqslant C]}{W;[G \mid a:\Box A \leqslant C]}\ [\Box_L^{\leqslant}(1)]$$

$$\frac{W \cup \{(c, a:\Box A)\};[G \mid a:\Diamond c \leqslant c:A \mid C < c:A] \qquad W;[G \mid C < \top]}{W;[G \mid C < a:\Box A]}\ [\Box_{R1}^{<}(2)]$$

$$\frac{W \cup \{(c, a:\Box A)\};[G \mid a:\Diamond c \leqslant c:A \mid C \leqslant c:A]}{W;[G \mid C \leqslant a:\Box A]}\ [\Box_{R1}^{\leqslant}(2)]$$

$$\frac{W \cup \{(c, a:\Box A)\};[G \mid a:\Diamond c \leqslant c:A \mid C < c:A] \qquad W \cup \{(c, a:\Box A)\};[G \mid C < \top]}{W \cup \{(c, a:\Box A)\};[G \mid C < a:\Box A]}\ [\Box_{R2}^{<}]$$

$$\frac{W \cup \{(c, a:\Box A)\};[G \mid a:\Diamond c \leqslant c:B \mid C \leqslant c:A]}{W \cup \{(c, a:\Box A)\};[G \mid C \leqslant a:\Box A]}\ [\Box_{R2}^{\leqslant}]$$

$$\frac{W \cup \{(c, a:\Box A)\};[G \mid a:\Diamond c \leqslant c:A \mid a:\Box A < c:A] \qquad W \cup \{(c, a:\Box A)\};[G \mid a:\Box A < \top]}{W \cup \{(c, a:\Box A)\};[G]}\ [W_{\Box}^{1}]$$

$$\frac{W \cup \{(c, a:\Box A)\};[G \mid c:A < a:\Diamond c] \qquad W \cup \{(c, a:\Box A)\};[G \mid c:A < a:\Box A]}{W \cup \{(c, a:\Box A)\};[G]}\ [W_{\Box}^{2}]$$

$$\frac{W \cup \{(c, a:\Diamond A)\};[G \mid a:\Diamond c \triangleleft C \mid c:A \triangleleft C]}{W;[G \mid a:\Diamond A \triangleleft C]}\ [\Diamond_{L1}(3)]$$

$$\frac{W \cup \{(c, a:\Diamond A)\};[G \mid a:\Diamond c \triangleleft C \mid c:A \triangleleft C]}{W \cup \{(c, a:\Diamond A)\};[G \mid a:\Diamond A \triangleleft C]}\ [\Diamond_{L2}]$$

$$\frac{W;[G \mid C \triangleleft a:\Diamond A \mid C \triangleleft a:\Diamond b] \qquad W;[G \mid C \triangleleft a:\Diamond A \mid C \triangleleft b:A]}{W;[G \mid C \triangleleft a:\Diamond A]}\ [\Diamond_R(4)]$$

$$\frac{W \cup \{(c, a:\Diamond A)\};[G \mid a:\Diamond A < a:\Diamond c] \qquad W \cup \{(c, a:\Diamond A)\};[G \mid a:\Diamond A < c:A]}{W \cup \{(c, a:\Diamond A)\};[G]}\ [W_{\Diamond}^{1}]$$

$$\frac{W \cup \{(c, a:\Diamond A)\};[G \mid a:\Diamond c < a:\Diamond A \mid c:A < a:\Diamond A]}{W \cup \{(c, a:\Diamond A)\};[G]}\ [W_{\Diamond}^{2}]$$

(1) $a:\Diamond b \in Form(W;[G \mid a:\Box A \triangleleft C])$.
(2) $c \notin Nom(W;[G \mid C \triangleleft a:\Box A])$ and there is no pair in W containing $a:\Box A$.
(3) $c \notin Nom(W;[G \mid a:\Diamond A \triangleleft C])$ and $A \notin \mathrm{Nom}$.
(4) $a:\Diamond b \in Form(G \mid C \triangleleft a:\Diamond A)$ and $B \notin \mathrm{Nom}$.

Fig. 3. Modal rules for GH_n and WGH_∞.

A sequent $\mathcal{S}$ is valid in $\mathcal{M} = (U, R, V)$, written $\mathcal{M} \vDash \mathcal{S}$, iff for all $\mathcal{M}$-assignment g, $\mathcal{M}, g \vDash \mathcal{S}$ holds. A sequent $\mathcal{S}$ is valid in GH_n (resp. WGH_∞), written $\mathsf{GH}_n \vDash \mathcal{S}$ (resp. $\mathsf{WGH}_\infty \vDash \mathcal{S}$), iff it is valid in every GH_n-Kripke model (resp. WGH_∞-Kripke model).

Proposition 4.3. *Let A be a formula and $\mathcal{M}$ be a model, $\mathcal{M} \vDash A$ iff $\mathcal{M} \vDash a:A$ where $a \notin Nom(A)$.*

Proof. For the if part, we suppose that $\mathcal{M} \nvDash A$. Thus, by assigning to a the world in M where A is not valid, we obtain $\mathcal{M} \nvDash a:A$. The only if part is trivial. □

From Proposition 4.3 and the definition of the validity of the relational sequents in GH_n (resp. WGH_∞), we deduce that a formula A is valid in GH_n (resp. WGH_∞) iff $\mathsf{GH}_n \vDash \emptyset; [\top \leqslant a:A]$ (resp. $\mathsf{WGH}_\infty \vDash \emptyset; [\top \leqslant a:A]$) where $a \notin Nom(A)$.

Considering a proof rule as composed of premises H_i with a conclusion C, it is *sound* if, for any instance of the rule, the validity of the premises H_i entails the validity of C. It is *strongly sound* if, for any instance of the rule and any model $\mathcal{M}$, if $\mathcal{M}$ is a model of all the H_i then it is a model of C. Moreover, a proof rule is *invertible* if, for any instance of the rule, the non-validity of at least one H_i entails the non-validity of C. It is *strongly invertible* if, for any instance of this rule and any modal model $\mathcal{M}$, if $\mathcal{M}$ is a countermodel of at least one of its premises then it is a countermodel of its conclusion. We can observe that strong invertibility implies invertibility.

We call *derivation* of a sequent $\mathcal{S}$ any tree labelled with sequents such that the root node is labelled with $\mathcal{S}$ and the labels at the immediate successors of a node n are the premises of a rule having the label at n as conclusion.

Proposition 4.4. *Let $\mathcal{M} = (W, R, V)$ be a model and g be an $\mathcal{M}$-assignment. Then, for all $a, c \in \mathrm{Nom}$, $V_g(g(a), \Diamond c) = R(g(a), g(c))$.*

Proof. We have $V_g(g(a), \Diamond c) = supr\{R(g(a), w') * V_g(w', c)\}$. Since $V_g(w', c) > 0 (= 1)$ iff $g(c) = w'$, we deduce that $V_g(g(a), \Diamond c) = R(g(a), g(c))$. □

$$\frac{W;[G \mid b:A \lhd C]}{W;[G \mid a:b:A \lhd C]}\,[:_L] \qquad \frac{W;[G \mid A \lhd b:B]}{W;[G \mid A \lhd a:b:B]}\,[:_R] \qquad \frac{W;[G \mid \top \lhd C]}{W;[G \mid a:a \lhd C]}\,[id_L]$$

$$\frac{W;[G \mid A \lhd \top]}{W;[G \mid A \lhd a:a]}\,[id_R] \qquad \frac{W;[G \mid \bot \lhd C]}{W;[G \mid a:\bot \lhd C]}\,[\bot_L] \qquad \frac{W;[G \mid A \lhd \bot]}{W;[G \mid A \lhd a:\bot]}\,[\bot_R]$$

$$\frac{W;[G \mid \top \lhd C]}{W;[G \mid a:\top \lhd C]}\,[\top_L] \qquad \frac{W;[G \mid A \lhd \top]}{W;[G \mid A \lhd a:\top]}\,[\top_R]$$

$$\frac{(W;[G \mid \top < C])[a/b] \qquad W;[G \mid \bot < C \mid \top \leqslant a:b]}{W;[G \mid a:b < C]}\,[sub^1_L] \qquad \frac{(W;[G \mid \top \leqslant C])[a/b]}{W;[G \mid a:b \leqslant C]}\,[sub^2_L]$$

$$\frac{(W;[G \mid A < \top])[a/b] \qquad W;[G \mid \top \leqslant a:b]}{W;[G \mid A < a:b]}\,[sub^1_R] \qquad \frac{W;[G \mid A \leqslant \bot \mid \top \leqslant a:b]}{W;[G \mid A \leqslant a:b]}\,[sub^2_R(*)]$$

$(*)$ $A \neq \top$.

Fig. 4. Satisfaction rules for GH_n and WGH_∞.

Proposition 4.5. *Let A be a formula, a and b be two nominals, $\mathcal{M} = (W, R, V)$ be a model and g be an $\mathcal{M}$-assignment. If $g(a) = g(b)$ then, for all $w \in W$, $V_g(w, A) = V_g(w, A[a/b])$.*

Proof. By structural induction on A. $\square$

Theorem 4.6. *The (logical, modal and satisfaction) rules of GH_n and WGH_∞ are strongly sound.*

Proof. We know that the GH_n-Kripke models and the WGH_∞-Kripke models are *witnessed* models: the infimum and the supremum functions, used in the definition of the extension of the valuations to formulas, correspond respectively to the minimum and the maximum. So using this common property, we deal with GH_n and WGH_∞ without distinguishing one from the others.

Let $L \in \{\mathsf{GH}_n \mid n \in \overline{\mathbb{N}}^* \text{ and } n \neq \infty\} \cup \{\mathsf{WGH}_\infty\}$. A rule $\frac{H_1 \cdots H_k}{C}$ $[R]$ is strongly sound in L iff if $\mathcal{M}$ is a countermodel of C then there exists $i \in \{1, \ldots, k\}$ such that $\mathcal{M}$ is a countermodel of H_i. We only develop the cases of $[\supset^<_L]$, $[\supset^\leqslant_R]$, $[\Box^<_L]$, $[\Box^\leqslant_R]$, $[W^1_\Box]$, $[W^2_\Box]$ and $[sub^1_L]$, the other cases being similar.

- Case $[\supset^<_L]$. Let $\mathcal{S} = W;[G \mid a : A \supset B < C]$, $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash \mathcal{S}$. Then, $\mathcal{M}, g \nvDash W;[G]$ and $V_g(g(a), A \supset B) \geqslant V_g(C)$ hold. If $V_g(g(a), A) \leqslant V_g(g(a), B)$ then $\mathcal{M}, g \nvDash a : B < a : A$ and we deduce $\mathcal{M}, g \nvDash W;[G \mid a : B < a : A]$. Otherwise, we have $(V_g(g(a), A) > V_g(g(a), B))$. Then, $V_g(g(a), A \supset B) = V_g(g(a), B)$ and $V_g(g(a), B) \geqslant V_g(C)$ hold. Thus, we have $\mathcal{M}, g \nvDash a : B < C$ and we deduce that $\mathcal{M}, g \nvDash W;[G \mid a : B < C]$.
- Case $[\supset^\leqslant_R]$. Let $\mathcal{S} = W;[G \mid C \leqslant a : A \supset B]$, $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash \mathcal{S}$. Then, $\mathcal{M}, g \nvDash W;[G]$ and $V_g(C) > V_g(g(a), A \supset B)$ hold. If $V_g(g(a), A) \leqslant V_g(g(a), B)$ then $V_g(g(a), A \supset B) = 1 < V_g(C)$ and we get a contradiction. Then $V_g(g(a), A) > V_g(g(a), B)$ and $V_g(C) > V_g(g(a), B) = V_g(g(a), A \supset B)$ hold. Thus, $\mathcal{M}, g \nvDash a : A \leqslant a : B \mid C \leqslant a : B$ holds and we deduce that $\mathcal{M}, g \nvDash W;[G \mid a : A \leqslant a : B \mid C \leqslant a : B]$.
- Case $[\Box^<_L]$. Let $\mathcal{S} = W;[G \mid a : \Box A < C]$ such that $a : \Diamond b \in \mathit{Form}(\mathcal{S})$, $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash \mathcal{S}$. Then, $\mathcal{M}, g \nvDash W;[G]$ and $V_g(g(a), \Box A) \geqslant V_g(C)$ hold. We have $V_g(g(a), \Box A) = \min\{R(g(a), w') \twoheadrightarrow V_g(w', A) \mid w' \in U\}$ ($\mathcal{M}$ is a witnessed model). Using Proposition 4.4, $V_g(g(a), \Diamond b) = R(g(a), g(b))$ holds. Thus, $V_g(g(a), \Box A) \leqslant R(g(a), g(b)) \twoheadrightarrow V_g(g(b), A) = V_g(g(a), \Diamond b \supset b : A)$ and $V_g(g(a), \Diamond b \supset b : A) \geqslant V_g(C)$ hold. So we obtain $\mathcal{M}, g \nvDash a : \Diamond b \supset (b : A) < C$. Using the same arguments of the case $[\supset^<_L]$, we deduce that $\mathcal{M}, g \nvDash W;[G \mid a : \Box A < C \mid b : A < a : \Diamond b]$ or $\mathcal{M}, g \nvDash W;[G \mid a : \Box A < C \mid b : A < C]$.
- Case $[\Box^\leqslant_{R1}]$. Let $\mathcal{S} = W;[G \mid C \leqslant a : \Box A]$, $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash \mathcal{S}$. Then, $\mathcal{M}, g \nvDash W;[G]$ and $V_g(C) > V_g(g(a), \Box A)$ hold. We have $V_g(C) > \min\{R(g(b), w') \twoheadrightarrow V_g(w', B) \mid w' \in U\}$. Then, there exists $w' \in U$ such that $V_g(g(a), \Box A) = R(g(b), w') \twoheadrightarrow V_g(w', B) < V_g(C)$. Let c be a new nominal ($c \notin \mathit{Nom}(\mathcal{S})$). We define a new $\mathcal{M}$-assignment g' as follows:

$$g'(n) = \begin{cases} w' & \text{if } n = c \\ g(n) & \text{otherwise} \end{cases}$$

  Clearly, we have $\mathcal{M}, g' \nvDash W \cup \{(c, a : \Box A)\};[G \mid C \leqslant a : \Diamond c \supset (c : A)]$. Using the same arguments of the case $[\supset^\leqslant_R]$, we deduce that $\mathcal{M}, g' \nvDash W \cup \{(c, a : \Box A)\};[G \mid a : \Diamond c \leqslant c : A \mid C \leqslant c : A]$.
- Case $[\Box^\leqslant_{R2}]$. Let $\mathcal{S} = W \cup \{(c, a : \Box A)\};[G \mid C \leqslant a : \Box A]$, $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash \mathcal{S}$. Then, $\mathcal{M}, g \nvDash W \cup \{(c, a : \Box A)\};[G]$ and $V_g(C) > V_g(g(a), \Box A)$ hold. Hence, $V_g(C) > R(g(a), g(c)) \twoheadrightarrow V_g(g(c), A) = V_g(g(a), \Diamond c \supset c : A)$ holds. Thus, $\mathcal{M}, g \nvDash W \cup \{(c, a : \Box A)\};[G \mid C \leqslant a : \Diamond c \supset (c : A)]$. Using the same arguments of the case $[\supset^\leqslant_R]$, we deduce that $\mathcal{M}, g \nvDash W \cup \{(c, a : \Box A)\};[G \mid a : \Diamond c \leqslant c : A \mid C \leqslant c : A]$.

– Cases $[W^1_\Box]$ and $[W^2_\Box]$. The strongly soundness of these two rules comes from the fact that if $\mathcal{M} = (U, R, V)$ satisfies the witness context w.r.t. g then, for all pairs $(c, a : \Box A)$ in the witnessed context, $V_g(a : \Box A) = V_g(a : \Diamond c \supset c : A)$ holds.
– Case $[sub^1_L]$. Let $\mathcal{S} = W; [G \mid a : b < C]$, $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash \mathcal{S}$. Then, $\mathcal{M}, g \nvDash W; [G]$ and $V_g(g(a), b) \geqslant V_g(C)$ hold. If $g(a) = g(b)$ then $V_g(g(a), b) = 1 \geqslant V_g(C)$ holds and using Proposition 4.5 we obtain $\mathcal{M}, g \nvDash (W; [G \mid \top < C])[a/b]$. Otherwise, from $g(a) \neq g(b)$ we obtain $1 > V_g(g(a), b) = 0$ and $0 \geqslant V_g(C)$. Therefore, we deduce that $\mathcal{M}, g \nvDash W; [G \mid \bot < C \mid \top \leqslant a : b]$. $\Box$

Theorem 4.7. *The (logical, modal and satisfaction) rules of* GH_n *and* WGH_∞ *are strongly invertible.*

Proof. For the same reasons as these given in the proof of Theorem 4.6, we deal with GH_n and WGH_∞ without distinguishing one from the others.

Let $L \in \{\mathsf{GH}_n \mid n \in \overline{\mathbb{N}}^* \text{ and } n \neq \infty\} \cup \{\mathsf{WGH}_\infty\}$. A rule $\frac{H_1 \cdots H_k}{C}$ $[R]$ is strongly invertible in L iff, for some $i \in \{1, \ldots, k\}$, if $\mathcal{M}$ is a countermodel of H_i then $\mathcal{M}$ is a countermodel of C. We only develop the cases of $[\supset^<_L]$, $[\supset^\leqslant_R]$, $[\Box^\leqslant_{R1}]$ and $[sub^1_L]$, the other cases being similar or trivial.

– Case $[\supset^<_L]$. Subcase of the left premise. Let $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash W; [G \mid a : B < a : A]$. Then, $\mathcal{M}, g \nvDash W; [G]$ and $V_g(g(a), A \supset B) = 1$ hold. Since $\mathcal{M}, g \nvDash \top < C$, $\mathcal{M}, g \nvDash a : A \supset B < C$ holds and we deduce that $\mathcal{M}, g \nvDash W; [G \mid a : A \supset B < C]$.
Subcase of the right premise. Let $\mathcal{M} = (W, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash W; [G \mid a : B < C]$. Since $V_g(g(a), B) \leqslant V_g(g(a), A \supset B)$, $\mathcal{M}, g \nvDash a : A \supset B < C$ holds and we deduce that $\mathcal{M}, g \nvDash W; [G \mid a : A \supset B < C]$.
– Case $[\supset^\leqslant_R]$. Let $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash W; [G \mid a : A \leqslant a : B \mid C \leqslant a : B]$. Then, $\mathcal{M}, g \nvDash W; [G]$, $V_g(g(a), A) > V_g(g(a), B)$ and $V_g(C) > V_g(g(a), B)$ hold. Since $V_g(g(a), A) > V_g(g(a), B)$, $V_g(g(a), A \supset B) = V_g(g(a), B)$ and $V_g(C) > V_g(g(a), A \supset B)$ hold. Thus, $\mathcal{M}, g \nvDash W; [G \mid C \leqslant a : A \supset B]$ holds.
– Case $[\Box^\leqslant_{R1}]$. Let $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash W \cup \{(c, a : \Box A)\}; [G \mid a : \Diamond c \leqslant c : A \mid C \leqslant c : A]$. Then, $\mathcal{M}, g \nvDash W; [G]$, $V_g(g(a), \Diamond c) > V_g(g(c), A)$ and $V_g(C) > V_g(g(c), A)$ hold.
We have $V_g(g(a), \Box A) = V(g(a), \Diamond c \supset (c : A))$. Since $V_g(g(a), \Diamond c) > V_g(g(c), A)$, $V(g(a), \Diamond c \supset (c : A)) = V_g(g(c), A)$ and $V_g(C) > V(g(a), \Box A)$ hold. Thus, we deduce that $\mathcal{M}, g \nvDash W; [G \mid C \leqslant a : \Box A]$.
– Case $[sub^1_L]$. Subcase of the left premise. Let $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash (W; [G \mid \top < C])[a/b]$. Then, $\mathcal{M}, g \nvDash (W; [G])[a/b]$ holds. We define a new $\mathcal{M}$-assignment g' as follows:

$$g'(n) = \begin{cases} g(b) & \text{if } n = a \\ g(n) & \text{otherwise} \end{cases}$$

We can show that $\mathcal{M}, g' \nvDash W; [G]$ and $V'_g(g'(a), b) = 1$. Thus, $\mathcal{M}, g' \nvDash W; [G \mid a : b < C]$ holds.
Subcase of the right premise. Let $\mathcal{M} = (U, R, V)$ be an L-Kripke model and g be an $\mathcal{M}$-assignment such that $\mathcal{M}, g \nvDash W; [G \mid \bot < C \mid \top \leqslant a : b]$. Then, $\mathcal{M}, g \nvDash W; [G]$, $V_g(C) = 0$ and $1 > V_g(g(a), b)$ hold. From $1 > V_g(g(a), b)$ we have $g(a) \neq g(b)$ and then $V_g(g(a), b) = 0$. Thus, $\mathcal{M}, g \nvDash W; [G \mid a : b < C]$ holds. $\Box$

5. Decidability and finite model property

In this section, we propose decision procedures with countermodel generation for the logics $(\mathsf{GH}_n)_{0<n<\infty}$ and WGH_∞. Our approach is based on two main steps:

1. to reduce, by a proof search process using our proof rules, a sequent into a set of so-called irreducible sequents;
2. to decide these irreducible sequents by a specific procedure.

The main problem is that the application of our proof rules to a sequent does not always terminate. In order to solve it we introduce a notion of *normal derivation* where the number of applications of the problematic rules is finite. The decision procedure for the irreducible sequents is obtained by using a technique developed for Gödel logics [22]. It consists of the construction of a semantic graph, called *bi-colored graph*, and of a characterization of validity based on the detection of particular chains in such a graph.

5.1. Termination and irreducible sequents

It is easy to see that the application of our proof rules to a sequent does not always terminate. This is mainly due to the rules $[\Box^<_L]$, $[\Box^\leqslant_L]$, $[\Diamond_R]$, $[W^1_\Box]$, $[W^2_\Box]$, $[W^1_\Diamond]$ and $[W^2_\Diamond]$ where in each one the premises are more complex than the conclusion. We call these rules the *contraction rules*. To solve this problem, we define particular derivations, called *normal derivations*, in which we avoid unnecessary applications of these rules.

Definition 5.1. A *normal derivation* is a derivation satisfying the following conditions in any branch:

- for the same pair of principal formulas $(a : \Box A, a : \Diamond b)$ and the same component $a : \Box A \lhd C$, the rule $[\Box_L^{\lhd}]$ is applied only once;
- for the same pair of principal formulas $(a : \Diamond A, a : \Diamond b)$ and the same component $C \lhd a : \Diamond A$, the rule $[\Diamond_R]$ is applied only once;
- for the same pair $(c, a : \Box A)$ in the witness context, the rule $[W^1_\Box]$ (resp. $[W^2_\Box]$) is applied once;
- for the same pair $(c, a : \Diamond A)$ in the witness context, the rule $[W^1_\Diamond]$ (resp. $[W^2_\Diamond]$) is applied once; and
- a branch is finite if and only if we cannot apply any non-contraction rule to the sequent of the last node of this branch.

Definition 5.2. Let A be a formula, the *subformulas* of A are defined by

- A is a subformula of A;
- if $B \otimes C$ is a subformula of A then so are B, C, for $\otimes = \wedge, \vee, \supset$;
- if $\boxtimes B$ is a subformula of A then so is B, for $\boxtimes = \Box, \Diamond$;
- if $a : B$ is a subformula of A then so is B.

In order to prove the finiteness of any normal derivation, we introduce a notion of subformula different from the usual one, called *N-subformula*, N being a set of nominals.

Definition 5.3. Let N be a finite set of nominals and $a : A$ be a formula. The *N-subformulas* of $a : A$ are defined as follows:

- $a : A$ is an N-subformula of $a : A$;
- if $b : B$ is an N-subformula of $a : A$ then so is $(b : B)[c_1/c_2]$, for $c_1, c_2 \in \mathrm{Nom}$ and $c_2 \in N$;
- if $b : B \otimes C$ is an N-subformula of $a : A$ then so are $b : B$ and $b : C$, for $\otimes = \wedge, \vee, \supset$;
- if $b : \boxtimes B$ is an N-subformula of $a : A$ then so is $c : B$, for $c \in \mathrm{Nom}$ and $\boxtimes = \Box, \Diamond$.

Proposition 5.4. *If $b : B$ is an N-subformula of $a : A$ then B is obtained from a subformula of $a : A$ by substituting some of its nominals (possibly none) by nominals in N.*

Proof. By structural induction on A. $\Box$

Theorem 5.5. *Let $\mathcal{S}$ be a sequent, $N = Nom(\mathcal{S})$ and $\mathcal{D}$ be a normal derivation of $\mathcal{S}$. Any formula appearing in $\mathcal{D}$ is either an N-subformula of a formula in $\mathcal{S}$, a formula of the form $a : \Diamond c$ or a constant.*

Proof. We only have to prove that for all the rules, if any formula appearing in the conclusion is either an N-subformula of a formula in $\mathcal{S}$, a formula of the form $a : \Diamond c$ or a constant then so is any formula in the premises. $\Box$

Definition 5.6. The *nesting degree* of a formula A, denoted $nest(A)$, is defined inductively as follows:

- $nest(p) = nest(a) = nest(\bot) = nest(\top) = 0$;
- $nest(A \otimes B) = \max(nest(A), nest(B))$ where $\otimes \in \{\wedge, \vee, \supset\}$;
- $nest(a : A) = nest(A)$; and
- $nest(\boxtimes A) = 1 + nest(A)$ where $\boxtimes \in \{\Box, \Diamond\}$.

Proposition 5.7. *Let $\mathcal{S}$ be a sequent and $\mathcal{D}$ be a normal derivation of $\mathcal{S}$. The set of the formulas of the form $a : \Diamond c$ appearing in $\mathcal{D}$ is finite.*

Proof. Let $N = Nom(\mathcal{S})$. We start by proving that for every nominal a, the set of the formulas $a : \Diamond c$ appearing in $\mathcal{D}$ is finite. For all $a : \Diamond c$ appearing in $\mathcal{D}$, $c \in N$ or c is introduced using one of the rules $[\Box_{R1}^{<}]$, $[\Box_{R1}^{\leqslant}]$ and $[\Diamond_{L1}]$ with a formula $a : \boxtimes A$ ($\boxtimes = \Box$ or $\Diamond$) as principal formula. We know that every formula $a : \boxtimes A$ introduces at most one nominal. Using Proposition 5.4 and Theorem 5.5, $\boxtimes A$ is obtained from a subformula of formula in $\mathcal{S}$ by substituting some of its nominals by nominals in N. Since N and the set of the subformulas of the formulas in $\mathcal{S}$ are finite, we deduce that for every nominal a, the set of the formulas $a : \Diamond c$ appearing in $\mathcal{D}$ is finite.

We can prove by induction on $k = \max\{nest(a : A) \mid a : A \in Form(\mathcal{S})\}$ that in every sequent appearing in $\mathcal{D}$, there is no chain of the form $a : \Diamond a_1, a_1 : \Diamond a_2, \ldots, a_{n-1} : \Diamond a_n$ with $a_1, \ldots, a_n \notin N$ satisfying $n > k$. Thus, we deduce that there is no chain $a_0 : \Diamond a_1, a_1 : \Diamond a_2, \ldots, a_{n-1} : \Diamond a_n$ where $n > nest(\mathcal{S}) \times \#N$ and for $i, j = 0, \ldots, n$, $a_i \neq a_j$ holds. Therefore, the set of the formulas $a : \Diamond c$ appearing in $\mathcal{D}$ is finite. Indeed, the formulas of the form $a : \Diamond c$ in any sequent appearing in $\mathcal{D}$ form a directed graph where the nodes and the arcs are respectively the nominals and the formulas of the form $a : \Diamond c$ appearing in this sequent. In this graph, the number of successors of every node which differ from it is bounded by a number n and every chain of distinct nodes has a length smaller than $k = nest(\mathcal{S}) \times \#N$. Such graphs have the number of nodes smaller than $\frac{n^{k+1}-1}{n-1}$ (the number of the nodes of the complete n-ary tree of height k). $\square$

Theorem 5.8. *All the normal derivations are finite.*

Proof. Using Proposition 5.7 and the conditions associated to the application of the contraction rules, we deduce that the number of applications of the modal rules in any normal derivation are finite. Thus, it is sufficient to prove that every derivation using only the logical and the satisfaction rules is finite. To do this, we show that for every rule, its conclusion is more complex than its premises by using a measure of complexity over the components. The first measure on formulas is defined by:

- $\alpha(p) = 1$ where $p \in \mathsf{Var} \cup \mathsf{Nom} \cup \{\bot, \top\}$;
- $\alpha(A \otimes B) = \alpha(A) + \alpha(B) + 1$ where $\otimes \in \{\wedge, \vee, \supset\}$;
- $\alpha(\boxtimes A) = \alpha(A) + 1$ where $\boxtimes \in \{\Box, \Diamond\}$;
- $\alpha(a : A) = 1 + \alpha(A)$.

From this definition we define a measure over the components:

$$\beta(A \lhd C) = \begin{cases} 0 & \text{if } A \equiv \top,\ \lhd = \leqslant \text{ and } C \equiv a : b \\ \alpha(A) + \alpha(B) & \text{otherwise} \end{cases}$$

The order relation $>$ on components, with $C_1 > C_2$ iff $\beta(C_1) > \beta(C_2)$, is well-founded. Now, we define an order relation on multisets of components: let M_1 and M_2 be two multisets of components, $M_1 >_m M_2$ iff M_2 is obtained form M_1 by replacing one or more components by a finite number of components, such that if C_1 is replaced by C_2 then $\beta(C_1) > \beta(C_2)$. Since the relation order on components is well-founded, the order relation $>_m$ is well-founded [11]. It is the order relation which is used to show that in every rule, the conclusion is greater than the premises. For the rule $[sub_R^2]$, we have $\beta(A \leqslant a : b) > \beta(A \leqslant \bot)$ and $\beta(A \leqslant a : b) > \beta(\top \leqslant a : b) = 0$. From this we deduce that for every instance of $[sub_R^2]$, the conclusion is more complex than the premise (according $>_m$). $\square$

Definition 5.9. An *irreducible sequent* is a sequent appearing in a leaf node of a normal derivation.

Proposition 5.10. *In any irreducible sequent, any component has one of the following forms*:

- $a : \Box A \lhd b : \Diamond B$ *where* $B \notin \mathsf{Nom}$;
- $a : \Box A \lhd b : p$ *where* $p \in \mathsf{Prop}$;
- $a : \Box A \lhd C$ *where* $C \in \{\bot, \top\}$;
- $a : p \lhd b : \Diamond A$ *where* $p \in \mathsf{Prop}$;
- $C \lhd \Diamond A$ *where* $C \in \{\bot, \top\}$;
- $C_1 \lhd C_2$ *where for* $i = 1, 2$, $C_i \in \{\bot, \top\}$, $C_i \equiv a : p$ *with* $p \in \mathsf{Prop}$ *or* $C_i \equiv a : \Diamond b$ *with* $a, b \in \mathsf{Nom}$;
- $\top \leqslant a : b$ *where* $a, b \in \mathsf{Nom}$.

Proof. There is always a rule that can be applied to any sequent containing a component that is not of one of the previous forms. $\square$

5.2. Decision procedures for irreducible sequents

We have seen that any normal derivation is finite and their leaf nodes are labelled with irreducible sequents. Hence, from the strong soundness and invertibility of our proof rules, if we have a decision procedure for the irreducible sequents then we can obtain a decision procedure for the sequents by combining the procedure for the irreducible sequents with a proof-search using our rules.

Let us introduce some concepts first developed for Gödel logics [22].

Definition 5.11. A *bi-colored graph* is a finite oriented graph with two kinds of arrows, the green ones represented by $\rightarrow$ and the red ones represented by $\Rightarrow$.

We use the symbols $\rightarrow$ and $\Rightarrow$ to denote the corresponding relation in the graph. For example $\rightarrow\Rightarrow$ represents the composition of two relations and $u \rightarrow\Rightarrow w$ means that there exists a path $u \rightarrow v \Rightarrow w$ in the graph. The relation $\rightarrow^\star$ is the reflexive and transitive closure of $\rightarrow$, i.e., the accessibility of the relation $\rightarrow$. Moreover $\rightarrow + \Rightarrow$ is the union of both relations.

Definition 5.12. Let $\mathcal{G}$ be a bi-colored graph, a $\Rightarrow$*-cycle* of $\mathcal{G}$ is a chain of the form $u(\rightarrow + \Rightarrow)^{\star} \Rightarrow u$ and a *k-alternating chain* of $\mathcal{G}$ is a chain of the form $(\rightarrow^{\star}\Rightarrow)^{k}$.

The key point of our approach consists in associating a bi-colored graph to a given irreducible sequent in the given logic and in relating validity with the existence of $\Rightarrow$-cycle or k-alternating chain. Let us consider this approach for our new logics.

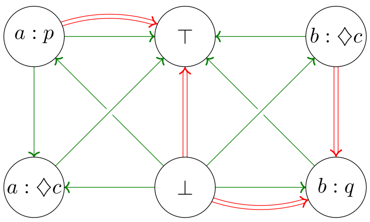
Definition 5.13. Let $\mathcal{S} = W; [Cp_1 \mid Cp_2 \mid \cdots \mid Cp_k]$ be an irreducible sequent, *the bi-colored graph* $\mathcal{G}_{\mathcal{S}}$ associated to $\mathcal{S}$ is built as follows:

- the set of the *nodes*, denoted $\mathcal{N}$, is the union of the following sets:
  - $\{a : p \mid p \in \mathsf{Prop}$ and $a : p$ is a formula in $\mathcal{S}\}$;
  - $\{a : \Diamond b \mid b \in \mathsf{Nom}$ and $a : \Diamond b$ is a formula in $\mathcal{S}\}$;
  - $\{\bot, \top\}$;
- the set of the *arrows*, denoted $\mathcal{A}$, is the union of the set B and the sets $A_{i=1,\ldots,k}$ defined as follows:
  - $B = \{\bot \rightarrow n \mid n \in \mathcal{N}$ and $n \neq \top\} \cup \{n \rightarrow \top \mid n \in \mathcal{N}$ and $n \neq \bot\} \cup \{\bot \Rightarrow \top\}$;
  - for all $i \in \{1, \ldots, k\}$, A_i is obtained from the component Cp_i as follows:
    if $Cp_i \equiv a : \Box A \lhd b : \Diamond B$ such that $B \notin \mathsf{Nom}$ then $A_i = \emptyset$;
    else if $Cp_i \equiv \top \lhd a : b$ then $A_i = \emptyset$;
    else if $Cp_i \equiv a : \Box A < C$ then $A_i = \emptyset$;
    else if $Cp_i \equiv C < a : \Diamond A$ then $A_i = \emptyset$;
    else if $Cp_i \equiv a : \Box A \leqslant C$ (C is in $\mathcal{N}$) then $A_i = \{C \Rightarrow \top\}$;
    else if $Cp_i \equiv C \leqslant a : \Diamond A$ such that $A \notin \mathsf{Nom}$ (C is in $\mathcal{N}$) then $A_i = \{\bot \Rightarrow C\}$;
    else if $Cp_i \equiv C_1 \lhd C_2$ (C_1 and C_2 are in $\mathcal{N}$) then if $\lhd = \leqslant$ then $A_i = \{C_2 \Rightarrow C_1\}$; else $A_i = \{C_2 \rightarrow C_1\}$.

Let us illustrate this construction with the following sequent $\mathcal{S}$:

$$\emptyset; \big[a : \Box(p \wedge q) \leqslant a : p \mid a : \Diamond c < a : p \mid b : q \leqslant b : \Diamond c \mid b : q \leqslant a : \Diamond(p \wedge q)\big]$$

The bi-colored graph associated to $\mathcal{S}$ is the following:



Proposition 5.14. *Let $\mathcal{S}$ be an irreducible sequent and $\mathcal{G}_{\mathcal{S}}$ be its associated bi-colored graph. Let $\mathcal{M} = (W, R, V)$ be a countermodel of $\mathcal{S}$ w.r.t. the $\mathcal{M}$-assignment g and $\mathcal{CH} = C_1 \rightarrow \cdots \rightarrow C_k \Rightarrow C_{k+1}$ be a chain in $\mathcal{G}_{\mathcal{H}}$. Then we have $V_g(C_1) \leqslant \cdots \leqslant V_g(C_k) < V_g(C_{k+1})$.*

Proof. For all $C_i \rightarrow C_{i+1}$ in $\mathcal{C}H$, we have either $C_{i+1} < C_i$ is a component in $\mathcal{S}$, $C_{i+1} = \top$ or $C_i = \bot$. Thus, since $\mathcal{M}$ is a countermodel of $\mathcal{S}$ w.r.t. g, we deduce that, for all $i \in \{1, \ldots, k-1\}$, $V_g(C_i) \leqslant V_g(C_{i+1})$.

Moreover, since the $\Rightarrow$ arrow $C_k \Rightarrow C_{k+1}$ is in $\mathcal{G}_{\mathcal{S}}$, we have either

- $C_{k+1} = \top$ and $a : \Box A \leqslant C_k$ is a component in $\mathcal{S}$;
- $C_k = \bot$ and $C_{k+1} \leqslant a : \Diamond A$ is a component in $\mathcal{S}$;
- $C_k = \bot$ and $C_{k+1} = \top$; or
- $C_{k+1} \leqslant C_k$ is a component in $\mathcal{S}$.

Hence, since $\mathcal{M}$ is a countermodel of $\mathcal{S}$ w.r.t. g, we deduce that $V_g(C_k) < V_g(C_{k+1})$. $\Box$

We can define, from a bi-colored graph $\mathcal{G}$, the notion of *bi-height* that is a function $h : \mathcal{G} \rightarrow \mathbb{N}$ such that for any $u, v \in \mathcal{G}$, if $u \rightarrow v \in \mathcal{G}$ then $h(u) \leqslant h(v)$ and if $u \Rightarrow v \in \mathcal{G}$ then $h(u) < h(v)$ [20,22]. Then a countermodel can be generated from $\mathcal{G}$ by using the following results: if a bi-colored graph $\mathcal{G}$ does not contain a $\Rightarrow$-cycle (resp. an n-alternating chain) then there exists a bi-height h (resp. that satisfies $h(v) < n$ for any node v in $\mathcal{G}$). Moreover we can decide if a graph contains or not a $\Rightarrow$-cycle and also compute the bi-height both in linear time. For more details, refer to [22].

Theorem 5.15. *An irreducible sequent $\mathcal{S}$ has a countermodel in* GH_n *if and only if its bi-colored graph $\mathcal{G}_{\mathcal{S}}$ does not contain an $(n+1)$-alternating chain.*

Proof. Let us call $\mathcal{D}$ the normal derivation containing $\mathcal{S}$.

First we prove the *if* part. Let $\mathcal{S}' = W; [Cp_1 \mid \ldots \mid Cp_k]$ be an irreducible sequent. We suppose that $\mathcal{G}_{\mathcal{S}} = (\mathcal{N}, \mathcal{A})$ does not contain a chain of the form $(\to^\star\Rightarrow)^{n+1}$. Then there exists a bi-height $h : \mathcal{G}_{\mathcal{S}} \to \{0, \ldots, n\}$. Let h' be the function from $\{a : p \mid a \in Nom(\mathcal{S}) \text{ and } p \in \mathsf{Prop}\} \cup \{a : \Diamond b \mid a, b \in Nom(\mathcal{S})\}$ to $\{0, \ldots, n\}$ defined using h as follows:

$$h'(A) = \begin{cases} n & \text{if } A \in \mathcal{N} \text{ and } h(A) = h(\top) \\ h(A) & \text{if } A \in \mathcal{N},\ h(A) \neq h(\top) \text{ and } h(A) \neq h(\bot) \\ 0 & \text{otherwise} \end{cases}$$

Then, we define the GH_n-Kripke model $M = (U, R, V)$ as follows:

- $U = Nom(\mathcal{S})$;
- for all $a, a' \in U$, $R(a, a') = \frac{h'(a:\Diamond a')}{n}$;
- for all $p \in \mathsf{Prop}$ and $a \in U$, $V(a, p) = \frac{h'(a:p)}{n}$.

Moreover, we define the $\mathcal{M}$-assignment g as follows:

$$g(a) = \begin{cases} a & \text{if } a \in Nom(\mathcal{S}) \\ b & \text{otherwise, such that } b \in U \end{cases}$$

Let $nest'$ be the function obtained by changing in the definition of the nest value in the case $\Diamond A$ as follows:

$$nest'(\Diamond A) = \begin{cases} 0 & \text{if } A \in \mathsf{Nom} \\ 1 + nest'(A) & \text{otherwise} \end{cases}$$

Now, we prove that $\mathcal{M}$ is a countermodel of $\mathcal{S}$, i.e., a countermodel of any component $Cp_{i\in\{1,,k\}}$ and satisfying W.

First, we prove that, for any $Cp_i \equiv C_1 \lhd C_2$, $V_g(C_1) \ntrianglelefteq V_g(C_2)$ holds.

For the case of a component $C_p = C_1 \lhd C_2$ where $nest'(C_1) = nest'(C_2) = 0$, the proof is simple. For example in the case of the components of the form $\top \leqslant a : b$ $(a \neq b)$, since $g(a) = a \neq b = g(b)$, $V_g(a : b) = 0$ holds and we deduce that $V_g(\top) > V_g(a : b)$.

Now, we deal with the other cases. We prove that, for all components $Cp' = C_1' \lhd C_2'$ in the branch of $\mathcal{D}$ containing $\mathcal{S}'$ (denoted $\mathcal{B}$), $V_g(C_1') \ntrianglelefteq V_g(C_2')$ holds. This is done by induction on $nest'(C_1') + nest'(C_2')$ and on the length of the subbranch delimited by the sequent containing C_p' and $\mathcal{S}'$, denoted $\mathcal{SB}$.

If $nest'(C_1') + nest'(C_2') = 0$ then the result is obtained by a simple induction on the length of $\mathcal{SB}$. The first statement (length of $\mathcal{SB}$ equals to 0) is the case treated previously where the components of $\mathcal{S}$ does not contain a formula of the forms $a : \Box A$ and $a : \Diamond A$ with $A \notin \mathsf{Nom}$. In the other statement the proof is obtained using the induction hypothesis and arguments similar to the ones given in the proof of Theorem 4.7.

If the length of $\mathcal{SB}$ is equal to 0 and $nest'(C_1') + nest'(C_2') > 0$ then we have the following cases:

- $Cp' = a : \Box A \lhd C$ where $C \equiv c : p$ or $C \in \{\top, \bot\}$;
- $Cp' = C \lhd a : \Diamond A$ where $C \equiv c : p$ or $C \in \{\top, \bot\}$ $(A \notin \mathsf{Nom})$;
- $Cp' = a : \Box A \lhd b : \Diamond B$.

We only develop the case $Cp' = a : \Box A \leqslant c : p$, the other cases being similar. From the definition of our model $\mathcal{M}$, we have $V_g(c : p) < 1$. If there is no formula of the form $a : \Diamond b$ in $\mathcal{S}'$ then $V_g(a : \Box A) = 1$ and we obtain $V_g(a : \Box A) > V_g(c : p)$. Otherwise, for all $a : \Diamond b$ in $\mathcal{S}'$, there is a sequent in $\mathcal{B}$ containing either $Cp_1' = b : A < a : \Diamond b$ or $Cp_2' = b : A \leqslant c : p$. Hence, by applying the induction hypothesis $(nest'(b : A) + nest'(a : \Diamond b) = nest'(b : A) + nest'(c : p) < nest'(a : \Box A) + nest'(c : p))$, we deduce that $V_g(a : \Box A) > V_g(c : p)$.

In the case where the length of $\mathcal{SB}$ and $nest'(C_1) + nest'(C_2)$ are strictly positive, the proof is obtained simply by applying the induction hypothesis.

Now, we prove that $\mathcal{M}$ w.r.t. g satisfies W. For all $(c, a : \Box A) \in W$, there is a sequent appearing in $\mathcal{B}$ containing either the component $a : \Box A < \top$ or the components $a : \Diamond c \leqslant c : A$ and $a : \Box A < c : A$ (the rule $[W_\Box^1]$). We know that for all components $Cp' = C_1' \lhd C_2'$ appearing in $\mathcal{B}$, $V_g(C_1') \ntrianglelefteq V_g(C_2')$ holds. Thus, we deduce that, for all $(c, a : \Box A) \in W$, $V_g(a : \Box A) \geqslant R(g(a), g(c)) \twoheadrightarrow V_g(c : A)$. Moreover, from the rule $[W_\Box^2]$ and using similar arguments, we have, for all $(c, a : \Box A) \in W$, $V_g(a : \Box A) \leqslant R(g(a), g(c)) \twoheadrightarrow V_g(c : A)$. Hence, for all $(c, a : \Box A) \in W$, $V_g(a : \Box A) = R(g(a), g(c)) \twoheadrightarrow V_g(c : A)$. The case of the pairs of the form $(c, a : \Diamond A)$ is similar (from the rules $[W_\Diamond^1]$ and $[W_\Diamond^2]$). Therefore, $\mathcal{M}$ w.r.t. g satisfies W.

We now prove the *only if* part. Let $\mathcal{M} = (U, R, V)$ w.r.t. g be a countermodel of $\mathcal{S}$. We suppose that there exists a chain of the form $(\to^\star\Rightarrow)^{n+1}$ in $\mathcal{G}_{\mathcal{S}}$: $C_0 \to^\star \Rightarrow C_1 \to^\star \Rightarrow C_2 \to^\star \Rightarrow \cdots \to^\star \Rightarrow C_n \to^\star \Rightarrow C_{n+1}$. Thus, by Proposition 5.14, there is a strictly increasing sequence of $n+2$ elements in the set of truth values. As this set does contain only $n+1$ elements, we get a contradiction. $\Box$

Theorem 5.16. *$\mathcal{G}_{\mathcal{S}}$ An irreducible sequent $\mathcal{S}$ has a countermodel in* WGH_∞ *if and only if its bi-colored graph $\mathcal{G}_{\mathcal{S}}$ does not contain a $\Rightarrow$-cycle.*

Proof. The proof is similar to the one of Theorem 5.15.

First we prove the *if* part. if $\mathcal{G}_{\mathcal{S}} = (\mathcal{N}, \mathcal{A})$ does not contain a $\Rightarrow$-cycle, we know that there exist $n \in \mathbb{N}$ and bi-height $h : \mathcal{G}_{\mathcal{S}} \to \{0, \dots, n\}$. Let h' be the function from $\{a : p \mid a \in Nom(\mathcal{S})$ and $p \in \mathsf{Prop}\} \cup \{a : \Diamond b \mid a, b \in Nom(\mathcal{S})\}$ to $\{0, \dots, n\}$ defined using h as follows:

$$h'(A) = \begin{cases} n & \text{if } A \in \mathcal{N} \text{ and } h(A) = h(\top) \\ h(A) & \text{if } A \in \mathcal{N},\ h(A) \neq h(\top) \text{ and } h(A) \neq h(\bot) \\ 0 & \text{otherwise} \end{cases}$$

Then, we define the WGH_∞-Kripke model $M = (U, R, V)$ as follows:

- $U = Nom(\mathcal{S})$;
- for all $a, a' \in U$, $R(a, a') = \frac{h'(a:\Diamond a')}{n}$;
- for all $p \in \mathsf{Prop}$ and $a \in U$, $V(a, p) = \frac{h'((w,p))}{n}$.

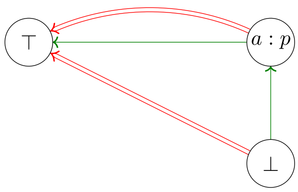
Moreover, we define the $\mathcal{M}$-assignment g as follows:

$$g(a) = \begin{cases} a & \text{if } a \in Nom(\mathcal{S}) \\ b & \text{otherwise, where } b \in U \end{cases}$$

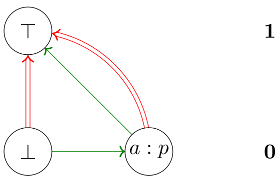
Similarly to the proof of Theorem 5.15, we prove that $\mathcal{M}$ is a countermodel of $\mathcal{S}$.

For the *only if* part: the existence of a chain $C \to^\star \Rightarrow \to^\star \cdots \to^\star \Rightarrow \to^\star \Rightarrow C$ implies that there exists $x \in [0, 1]$ such that $x < x$ (Proposition 5.14) and then we get a contradiction. □

Let us give a simple example: $\mathcal{S} = \emptyset; [a : \Box p \leqslant a : p]$. It is easy to see that we cannot apply any rule to $\mathcal{S}$. Thus, $\mathcal{S}$ is an irreducible sequent. Its associated bi-colored graph is:



This graph does not contain a $\Rightarrow$-cycle. In order to extract a countermodel we modify the previous graph in such a way that $\Rightarrow$ arrows always go up and $\to$ arrows never go down.



Then, we deduce that $\mathcal{M} = (\{a\}, R, V)$, where $R(a, a) = 0$ and $V(a, p) = 0$, is a countermodel of $\mathcal{S}$ in GH_1 and then in $(\mathsf{GH}_n)_{0<n<\infty}$ and WGH_∞.

Let us recall that the search for alternating chains in bi-graphs in the case of Gödel logics can be characterized as a resource use bounding problem in some particular process calculus [21]. Intuitively, the red arrows represent ressource consumption and this allows to characterize Gödel logics as resource use bounding logics. In future work, we will study the possibility to establish a similar characterization for Gödel hybrid logics.

5.3. *Decision procedures for* $(\mathsf{GH}_n)_{0<n<\infty}$ *and* WGH_∞

Now, we illustrate the results previously obtained by proposing decision procedures with countermodel generation for L with $L \in \{(\mathsf{GH}_n)_{0<n<\infty}, \mathsf{WGH}_\infty\}$. The main point is that for every sequent $\mathcal{S}$ one can effectively find a set $\mathcal{IR}$ of irreducible sequents, so that $\mathcal{S}$ is valid in L iff $\mathcal{S}'$ is valid in L for every $\mathcal{S}' \in \mathcal{IR}$.

To decide a sequent $\mathcal{S}$ in WGH_∞ (resp. $(\mathsf{GH}_n)_{0<n<\infty}$), the steps of the procedure are the following:

Step 1: We generate from $\mathcal{S}$ the set $\mathcal{IR}$ of irreducible sequents obtained by application of our proof rules.

Step 2: For each irreducible sequent $\mathcal{S}'$ in $\mathcal{IR}$, we construct its associated bi-colored graph $\mathcal{G}_{\mathcal{S}'}$.

– If $\mathcal{G}_{\mathcal{S}'}$ contains a $\Rightarrow$-cycle (resp. an $(n+1)$-alternating chain) then $\mathcal{S}'$ is valid in WGH_∞ (resp. $(\mathsf{GH}_n)_{0<n<\infty}$).
– Otherwise, $\mathcal{S}'$ has a countermodel in WGH_∞ (resp. $(\mathsf{GH}_n)_{0<n<\infty}$) obtained from $\mathcal{G}_{\mathcal{S}'}$ as follows:

- We modify $\mathcal{G}_{\mathcal{S}'} = (\mathcal{N}, \mathcal{A})$ in such a way that $\Rightarrow$ arrows always go up and $\rightarrow$ arrows never go down. This is possible because $\mathcal{G}_{\mathcal{S}'}$ does not contain a $\Rightarrow$-cycle.
- We extract from this modified graph the bi-height h which is a function from the nodes of $\mathcal{G}_{\mathcal{S}'}$ to $\{0, \dots, n\}$ where
  · for all $C_1 \rightarrow C_2$ in $\mathcal{G}_{\mathcal{S}'}$, $h(C_1) \leqslant h(C_2)$;
  · for all $C_1 \Rightarrow C_2$ in $\mathcal{G}_{\mathcal{S}'}$, $h(C_1) < h(C_2)$.
  An efficient algorithm to compute a bi-height when there is no $\Rightarrow$-cycle is provided in [22].
- We define from h a new function $h' : \{a : p \mid a \in \mathit{Nom}(\mathcal{S}) \text{ and } p \in \mathsf{Prop}\} \cup \{a : \Diamond b \mid a, b \in \mathit{Nom}(\mathcal{S})\} \rightarrow \{0, \dots, n\}$ by:

$$h'(A) = \begin{cases} n & \text{if } A \in \mathcal{N} \text{ and } h(A) = h(\top) \\ h(A) & \text{if } A \in \mathcal{N},\ h(A) \neq h(\top) \text{ and } h(A) \neq h(\bot) \\ 0 & \text{otherwise} \end{cases}$$

The countermodel $\mathcal{M} = (W, R, V)$ of $\mathcal{S}'$ in WGH_∞ (resp. $(\mathsf{GH}_n)_{0<n<\infty}$) is obtained as follows:

- $W = \mathit{Nom}(\mathcal{S}')$;
- for all $a, a' \in W$, $R(a, a') = \frac{h'(a:\Diamond a')}{n}$;
- for all $p \in \mathsf{Prop}$ and $a \in W$, $V(a, p) = \frac{h'(a:p)}{n}$.

If all the elements of $\mathcal{IR}$ are valid then, from soundness of our proof rules, we obtain that $\mathcal{S}$ is valid. Otherwise there is an irreducible sequent in $\mathcal{IR}$ that has a countermodel $\mathcal{M}$. Hence, from the strong invertibility of our proof rules, $\mathcal{M}$ is a countermodel of $\mathcal{S}$.

Theorem 5.17. WGH_∞ *and* $(\mathsf{GH}_n)_{0<n<\infty}$ *are decidable and have the finite model property.*

Proof. From the above procedure and the way to build countermodels. □

The key point in our decision procedures is the fact that the infimum and the supremum correspond respectively to the minimum and the maximum. Since it is not the case for GH_∞, the same idea cannot be applied for this logic.

6. A sequent calculus for WGH_∞

In this section we propose a calculus for WGH_∞ called *WGH*, that is defined by the proof rules of Figs. 2–4 and the following axiom and structural rules:

$$\frac{}{W;[G \mid A \leqslant A]}\,[ID] \qquad \frac{W;[G \mid \top \leqslant \bot]}{W;[G]}\,[EW_1] \qquad \frac{W;[G \mid A < \bot]}{W;[G]}\,[EW_2]$$
$$\frac{W;[G \mid \top < A]}{W;[G]}\,[EW_3] \qquad \frac{W;[G \mid A \leqslant C]}{W;[G \mid A \leqslant B \mid B \triangleleft C]}\,[com_1] \qquad \frac{W;[G \mid A \leqslant C]}{W;[G \mid A \triangleleft B \mid B \leqslant C]}\,[com_2]$$

where in the rules $[EW_2]$ and $[EW_3]$, $A \in \mathit{Form}(G)$. However, this condition is not necessary.

Theorem 6.1 *(Soundness). The rules of WGH calculus are sound.*

Proof. From Theorem 4.6, the rules of Figs. 2–4 are sound. Similar arguments are used for the other rules. □

Theorem 6.2 *(Completeness). If a sequent is valid in* WGH_∞ *then it is derivable in WGH calculus.*

Proof. We only have to prove that, for any irreducible sequent, if its associated bi-colored graph contains a $\Rightarrow$-cycle then it has a derivation using only the structural rules and the axiom. We know that any arrow in the bi-colored graphs correspond to an inequality: $A \Rightarrow B$ and $A \rightarrow B$ correspond respectively to $B \leqslant A$ and $B < A$.

Now, we prove that for any irreducible sequent $\mathcal{S}$, we can obtain, using the structural rules, a sequent containing all the components associated to the arrows of $\mathcal{G}_{\mathcal{S}}$.

– Case of $C \Rightarrow \top$ obtained from the component $a : \Box A \leqslant C$ in $\mathcal{S}$. Since any component $a : \Box A \leqslant C$ introduces a unique arrow, it is sufficient to replace it by $\top \leqslant C$:

$$\dfrac{\dfrac{W;[G \mid \top \leqslant C]}{W;[G \mid \top < a:\Box A \mid a:\Box A \leqslant C]}\,[com_2]}{W;[G \mid a:\Box A \leqslant C]}\,[EW_3]$$

– Case of $\perp \Rightarrow C$ obtained from the component $C \leqslant a : \Diamond A$ in $\mathcal{S}$:

$$\dfrac{\dfrac{W; [G \mid C \leqslant \perp]}{W; [G \mid C \leqslant a : \Diamond A \mid a : \Diamond A < \perp]}\,[com_1]}{W; [G \mid C \leqslant a : \Diamond A]}\,[EW_2]$$

The components associated to the arrows of the set B are obtained directly by using $[EW_1]$, $[EW_2]$ and $[EW_3]$. The components associated to the other arrows of $\mathcal{G}_\mathcal{S}$ are in $\mathcal{S}$. Clearly, if $\mathcal{G}_\mathcal{S}$ contains a $\Rightarrow$-cycle then by a succession of applications of the rules $[com_1]$ and $[com_2]$ to the sequent containing all the components associated to the arrows of $\mathcal{G}_\mathcal{S}$, we obtain an axiom. □

We conjecture that we can obtain systems for $(\mathsf{GH}_n)_{0<n<\infty}$ by only changing $[ID]$ in *WGH* with a more general axiom. This will be studied in further work.

7. Conclusion

In this work, we propose a family of fuzzy hybrid logics based on Gödel logic and its finitary versions. They are obtained using an approach similar to the one used to introduce intuitionistic hybrid logic [8]. The key point consists in replacing classical logic in classical hybrid logic with Gödel logic and its finitary versions. This family is composed of two infinite-valued hybrid logics, namely GH_∞ and WGH_∞, and a sequence of finite-valued logics $(\mathsf{GH}_n)_{0<n<\infty}$. For WGH_∞ and $(\mathsf{GH}_n)_{0<n<\infty}$ we provide decision procedures with countermodel generation. A key point is the use of strongly invertible rules and consequently the ability to generate countermodels. Using these decision procedures we show that WGH_∞ and $(\mathsf{GH}_n)_{0<n<\infty}$ are decidable and have the finite model property. Another result is the definition of a sequent calculus for WGH_∞. In further works we will study proof-search in GH_∞ and also introduce other family of fuzzy hybrid logics based on other fuzzy logics like *Łukasiewicz logic* and *Product logic* by exploring their proof-theoretical properties.

References

[1] C. Areces, B. ten Cate, Hybrid logics, in: P. Blackburn, F. Wolter, J. van Benthem (Eds.), Handbook of Modal Logics, Elsevier, 2006.
[2] A. Avellone, M. Ferrari, P. Miglioli, Duplication-free tableau calculi and related cut-free sequent calculi for the interpolable propositional intermediate logics, Log. J. IGPL 7 (1999) 447–480.
[3] A. Avron, A constructive analysis of RM, J. Symbolic Logic 52 (1987) 939–951.
[4] A. Avron, A tableau system for Gödel–Dummett logic based on a hypersequent calculus, in: Int. Conference on Analytic Tableaux and Related Methods, TABLEAUX 2000, St Andrews, Scotland, in: Lecture Notes in Artificial Intelligence, vol. 1847, 2000, pp. 98–111.
[5] M. Baaz, A. Ciabattoni, C.G. Fermüller, Cut-elimination in a sequents-of-relations calculus for Gödel logic, in: ISMVL, 2001, pp. 181–186.
[6] P. Blackburn, Internalizing labelled deduction, J. Logic Comput. 10 (2000) 137–168.
[7] P. Blackburn, M. de Rijke, Y. Venema, Modal Logic, Cambridge University Press, 2001.
[8] T. Braüner, V. de Paiva, Intuitionistic hybrid logic, J. Appl. Log. 4 (2006) 231–255.
[9] X. Caicedo, R. Rodríguez, A Gödel modal logic, manuscript.
[10] A. Chagrov, M. Zakharyaschev, Modal Logic, Oxford University Press, 1996.
[11] N. Dershowitz, Z. Manna, Proving termination with multiset orderings, in: ICALP, in: Lecture Notes in Comput. Sci., vol. 71, 1979, pp. 188–202.
[12] M. Dummett, A propositional calculus with a denumerable matrix, J. Symbolic Logic 24 (1959) 96–107.
[13] M. Fitting, Many-valued modal logics, Fund. Inform. 15 (1991) 235–254.
[14] M. Fitting, Many-valued modal logics II, Fund. Inform. 17 (1992) 55–73.
[15] D. Galmiche, D. Larchey-Wendling, Y. Salhi, Provability and countermodels in Gödel–Dummett logics, in: Int. Workshop on Disproving: Non-Theorems, Non-Validity, Non-Provability, DISPROVING'07, Bremen, Germany, 2007, pp. 35–52.
[16] G.D. Giacomo, Decidability of class-based knowledge representation formalisms, PhD thesis, Universitá di Roma "La Sapienza", 1995.
[17] P. Hájek, Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, 1998.
[18] P. Hájek, Making fuzzy description logic more general, Fuzzy Sets and Systems 154 (2005) 1–15.
[19] J. Hansen, T. Bolander, T. Braüner, Many-valued hybrid logic, in: Advances in Modal Logic, 2008, pp. 111–132.
[20] D. Larchey-Wendling, Counter-model search in Gödel–Dummett logics, in: 2nd Int. Joint Conference IJCAR 2004, Cork, Ireland, in: Lecture Notes in Artificial Intelligence, vol. 3097, 2004, pp. 274–288.
[21] D. Larchey-Wendling, Bounding resource consumption with Gödel–Dummett logics, in: Int. Conference on Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2005, in: Lecture Notes in Artificial Intelligence, vol. 3835, 2005, pp. 682–696.
[22] D. Larchey-Wendling, Graph-based decision for Gödel–Dummett logics, J. Automat. Reason. 38 (2007) 201–225.
[23] G. Metcalfe, N. Olivetti, Proof systems for a Gödel Modal Logic, in: Int. Conference on Analytic Tableaux and Related Methods, TABLEAUX 2009, Oslo, Norway, in: Lecture Notes in Artificial Intelligence, 2009, pp. 682–696.
[24] G. Priest, Many-valued modal logics: a simple approach, RSL 1 (2008) 190–203.